Dynamic Strain Response of an Infinite Beam and Inverse Calculation of Impact Force by Numerical Laplace Transform

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A general method for determining the dynamic strain response of an infinite beam to impact force is presented. The method consists of formulating and solving the dynamic problem in the Laplace transform domain and obtaining the strain response by numerical inversion of the transformed solution. Also, an inverse method for estimating the impact force on an infinite beam is investigated. Once the strain is known, it is shown how the impact force can be reconstructed.

Key Words : Dynamic Response, Impact Force, Inverse Problem, Infinite Beam, Numerical Laplace Transform

1. Introduction

There has been a long-lived interest in elastic flexural wave propagation in an infinite beam and the associated problem of strain response of the beam due to a suddenly applied impact force. McGhie (1990) solved the Bernoulli-Euler beam equation using an impulse response function, and a wave motion solution of the flexural response of an infinite length beam was presented. Doyle (1986, 1989) solved a force-displacement relation for the Bernoulli-Euler beam in the frequency domain using the Fourier transform approach. The strain response is obtained by numerical inversion of the transformed solution. Doyle (1984a) established a force-strain relation for the Timoshenko beam in the frequency domain using the Fourier transform approach. And, inversion by use of a Fourier transform algorithm (Paz, 1985) was shown to allow the impact force history to be determined. The major advantage of using this transform approach is that the convolution and deconvolution processes are simply performed by multiplication and division, respectively. The inverse problem is much more difficult to solve than the direct problem where one is concerned with finding the response of a system to a given input or excitation. In the most general inverse problem, the response is known, but either the equations describing the process are unknown or the inputs are unknown. Typically, the inverse problem arises because measurements can only be made in easily accessible locations or perhaps a state variable can only be measured indirectly.

This paper derives a force-strain relation for the Bernoulli-Euler beam using the Laplace transform approach. The strain response and the impact force history are obtained by numerical inversion of the transformed solution using a Laplace transform algorithm (Krings et al., 1979 ; Im et al., 1994). The Laplace transform approach is better than the Fourier transform approach in that it can obtain the exact strain response in the direct problem. The resulting strain responses are compared to those of Doyle (1986).

2. Force-Strain Relation

Doyle (1984b) solved a force-displacement relation in an explicit form. The structure is

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assumed to be a narrow beam of the Bernoulli-Euler type. The applied force and the beam displacement are related by

$$EI\frac{\partial^4 v}{\partial x^4}+\rho A\frac{\partial^2 v}{\partial t^2}=F(t) \tag{1}$$

where EI and ρA are the stiffness and mass per unit length of the beam, respectively. $F(t)$ is the applied force, v is the displacement of the beam and x is the distance from the point of impact. The boundaries are considered to be at infinity. The force-strain relation becomes

$$e(x,\ t)=-Q_1\int_0^t \frac{F(\tau)}{\sqrt{(t-\tau)}}\sin\left[\frac{x^2}{Q_2(t-\tau)}-\frac{\pi}{4}\right]d\tau \tag{2}$$

where

$$Q_1=\frac{h}{4EI\sqrt{\pi}}\left(\frac{EI}{\rho A}\right)^{1/4},$$
$$Q_2=4\left(\frac{EI}{\rho A}\right)^{1/2}.$$

Q_1 and Q_2 depend only on the material and section properties of the beam. $e(x,\ t)$ is the strain at an arbitrary position x and h is the thickness of the beam.

Doyle (1986) solved a force-displacement relation for the Bernoulli-Euler beam in the frequency domain using the Fourier transform approach. The spectral relation between force and strain is

$$\hat{e}(x,\ \omega)=\frac{\hat{F}\cdot h}{8EIk}[e^{-kx}-ie^{-ikx}] \tag{3}$$

where

$$k=\sqrt{\omega}\cdot\left[\frac{\rho A}{EI}\right]^{1/4},$$

$\hat{e}$ and $\hat{F}$ are the strain spectrum and the force spectrum, respectively, and ω is the frequency.

A new force-strain relation for the Bernoulli-Euler beam is developed using the Laplace transform approach. Consider an infinite beam shown in Fig. 1.

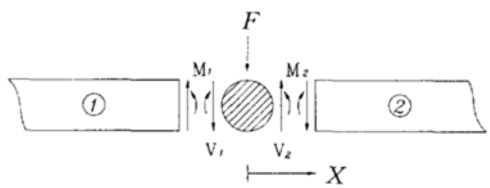


Fig. 1 Free body diagram for an infinite beam

The beam model assumes that only bending moment M and shear force V resultants act at the beam section. For ease of modeling, the force is assumed to act at a massless joint. Application of the Laplace transform with respect to time to Eq. (1) under zero initial conditions yields

$$\frac{d^4\bar{v}}{dx^4}+4K^4\bar{v}=\bar{F} \tag{4}$$

where

$$K=\sqrt{\frac{s}{2}}\cdot\left[\frac{\rho A}{EI}\right]^{1/4}$$

ρ is the mass per unit volume, A is the cross-sectional area, E is the Young's modulus, I is the moment of inertia of the cross section about the neutral axis and s is the Laplace transform parameter. The homogeneous solution of Eq. (4) takes the following form (Beskos et al., 1975)

$$\bar{v}(x,\ s)=e^{Kx}(A\cos Kx+B\sin Kx)$$
$$1+e^{-Kx}(C\cos Kx+D\sin Kx) \tag{5}$$

where A, B, C and D are constants. Since the beam is infinite then only two waves are generated.

$$x<0,\ \bar{v}_{(1)}=e^{Kx}(A\cos Kx+B\sin Kx) \tag{6}$$
$$x>0,\ \bar{v}_{(2)}=e^{-Kx}(C\cos Kx+D\sin Kx) \tag{7}$$

Imposing continuity of displacement and slope at the joint, and writing the equations of motion of the joint itself gives (at $x=0$)

Displacement: $\bar{v}_{(1)}=\bar{v}_{(2)}$

Slope: $\dfrac{d\bar{v}_{(1)}}{dx}=\dfrac{d\bar{v}_{(2)}}{dx}$

Moment: $EI\dfrac{d^2\bar{v}_{(1)}}{dx^2}=EI\dfrac{d^2\bar{v}_{(2)}}{dx^2}$

Shear Force: $-EI\dfrac{d^3\bar{v}_{(1)}}{dx^3}=-EI\dfrac{d^3\bar{v}_{(2)}}{dx^3}-\bar{F}$ (8)

Substituting Eqs. (6) and (7) into Eq. (8) gives the displacement and the strain for the positive x direction ($x>0$) as

$$\bar{v}_{(2)}=\frac{-e^{-Kx}(\cos Kx+\sin Kx)}{8EIK^3}\bar{F} \tag{9}$$

$$\bar{e}_{(2)}=\frac{-e^{-Kx}(\cos Kx-\sin Kx)\,h}{8EIK}\bar{F} \tag{10}$$

When the force spectrum $\bar{F}$ is known, then the strain spectrum (and hence the strain history) at an arbitrary position x can be obtained. The transfer function for the strain is

$$H(x, s)=\frac{-e^{-Kx}(\cos Kx-\sin Kx)h}{8EIK} \tag{11}$$

The advantage of Eq. (10) is that the transformed force is obtained by solving an algebraic equation. If a strain history is measured at an arbitrary position x, then the force history causing it can also be reconstructed by

$$\bar{F}=\frac{8EIK}{-e^{-Kx}(\cos Kx-\sin Kx)h}\bar{e}_{(2)} \tag{12}$$

This is the fundamental relation for obtaining the impact force from the strain. Eqs. (10) and (12) correspond to the following two cases, respectively: (i) given the force, the strain at an arbitrary position x can be computed by Eq. (10), (ii) given the strain at a position x, the force can be reconstructed by Eq. (12). Of course, once the Laplace transform components are obtained, the time histories are obtained simply by using the inverse Laplace transform. In this case, the force is obtained at the center of the beam ($x=0$).

3. Computer Implementation

A numerical Laplace transform algorithm can be used to conveniently convert a time function into its frequency components. Krings and Waller (1979) proposed a method of numerical Laplace transform using the algorithm of the Fast Fourier transform (FFT). The equations defining the Laplace transform and its inverse transform with $v(t)=0$ for $t<0$ are

$$\bar{v}(s)=\int_{t}^{\infty}v(t)e^{-st}dt$$
$$v(t)=\frac{1}{2\pi i}\int_{s=\gamma-i\infty}^{\gamma-i\infty}\bar{v}(s)e^{st}ds \tag{13}$$

where

$$s=\gamma+i\omega$$

Using the algorithm of the FFT, it is favourable to integrate along lines parallel to the imaginary axis, i. e. γ is constant, and it follows that $ds=i\,d\omega$. Then the Laplace transform can be converted into the Fourier transform

$$\bar{v}(s)=\int_{t=0}^{\infty}v(t)e^{-\gamma t}e^{-i\omega t}dt$$
$$=\int_{t=0}^{\infty}v^{*}(t)e^{-i\omega t}dt$$
$$=FT[v^{*}(t)] \tag{14}$$

$$v(t)=\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\bar{v}(s)e^{\gamma t}e^{i\omega t}d\omega$$
$$=e^{\gamma t}\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\bar{v}(s)e^{i\omega t}d\omega$$
$$=e^{\gamma t}FT^{-1}[\bar{v}(s)] \tag{15}$$

where FT[⋯] is the Fourier transform operator. For the evaluation of the Fourier integrals with the digital computer, discrete Fourier transform is used. It results from the continuous formula if only a finite time interval T is considered. This time internal is divided into N equal-time segments and the integration is performed by using the Euler formula

$$\bar{v}(\omega_j)=\left(\frac{T}{N}\right)\sum_{k=0}^{N-1}[v(t_k)e^{-\gamma t_k}]e^{-i\omega_j t_k} \tag{16}$$

$$v(t_k)=e^{\gamma t_k}\left(\frac{1}{T}\right)\sum_{j=0}^{N-1}[\bar{v}(\omega_j)e^{i\omega_j t_k}] \tag{17}$$

where j, $k=0, 1, 2, \cdots, N-1$

For example, the force history can be represented by

$$F(t_k)=e^{\gamma t_k}\left(\frac{1}{T}\right)\sum_{j=0}^{N-1}[\bar{F}(\omega_j)e^{i\omega_j t_k}] \tag{18}$$

where $\bar{F}(\omega_j)$ are the Laplace transform components at the discrete frequencies ω_j. It is suggested that one should select γT from 3 to 10 for good results (Krings et al., 1979 ; Im et al., 1994). The value of $\gamma T=6$ has been used in this work. If only strain for the forward wave is considered then Eqs. (10) and (12) can be written as follows :

$$\bar{e}(x, \omega_j)=H(x, \omega_j)\cdot\bar{F}(\omega_j) \tag{19}$$

$$\bar{F}(\omega_j)=\frac{\bar{e}(x, \omega_j)}{H(x, \omega_j)} \tag{20}$$

4. Numerical Example

Following numerical examples serve to illustrate the Laplace transform approach for determining the dynamic strain response of an infinite beam and the inverse method in estimating the impact force. All the numerical computations were performed by a personal computer. For the beam analysis, material properties and beam dimensions are assumed as nominal steel (see Table 1).

To illustrate some of the problems that arise in the numerical computations, the force shown in Fig. 2 is used as an input at the origin to obtain the strain history at various locations.

Figure 3 shows the effect of the window sizes on the Laplace transform approach and the Fourier transform approach (Doyle, 1986).

Strain histories at x=0 are computed with a sampling time of $10\mu s$. In Fig. 3, N is the number of the sampling points. If the same sampling rate is kept, then the window size can only be increased by increasing the number of sampling points. For this particular beam problem, a non-zero value of strain should persist for all time. However, in Fig. 3(a), the finite window size forces the strain to the initial value too soon. The distorted result of Fig. 3(a) is due to the singular point in Eq. (3). In order to meet the initial condition of strain history, the result of Fig. 3(a) should be corrected. Figure 3(b) shows the result of the Laplace transform approach which gives better results than the Fourier transform approach in the whole window size.

Figure 4 shows the strain histories at locations $x/h=0$, 100 and 200.

Once the transfer function at any location is known the strain history at the location is computed by Eq. (19). To compare the accuracy of Laplace transform approach with that of Fourier transform approach (Doyle, 1986), the strain histories using the Fourier transform approach are obtained by Eq. (3). On the other hand the solution of Eq. (2) is used as the exact one at x=0. But, at an arbitrary position x, the force-strain relation of Eq. (2) cannot be obtained in a closed form and therefore an approximate procedure (Doyle, 1984b) is used. Consider the force history to be made up of piece-wise constant segments. The integral can be replaced by a summation as follows :

Table 1 Material properties and parameters used in the examples

Property/Parameter	Value
E	210 GPa
ρ	7.86 g/cm^3
Width	5 mm
Thickness	1 mm

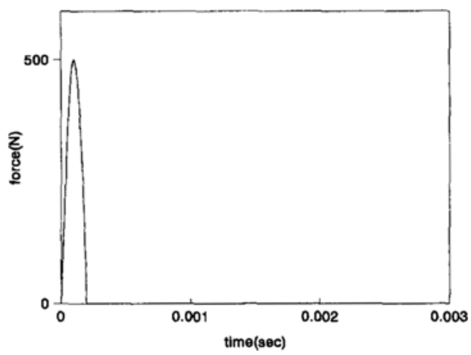
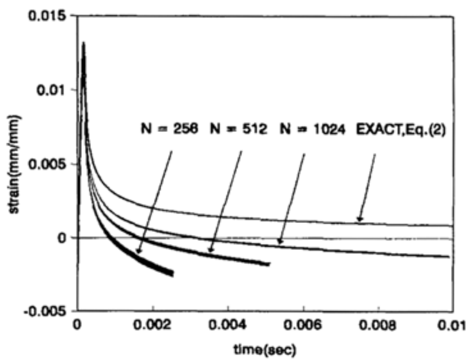
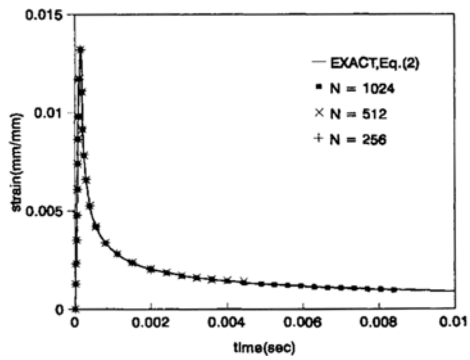


Fig. 2 A sample force history

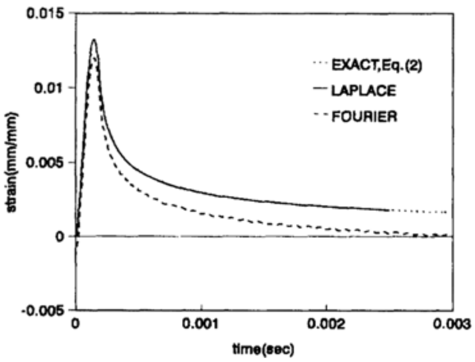


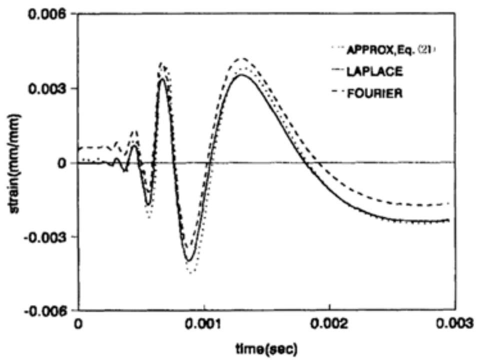
(a) Fourier transform approach



(b) Laplace transform approach

Fig. 3 The sensitivity of the computed strain history to the window size





(c) x/h=200

Fig. 4 Strain histories at different monitoring positions

$$e(x, t)=\sum_n[P(x, t, \tau_{n+1})-P(x, t, \tau_n)]\,F(\tau_{n+1/2}) \qquad (21)$$

where P is a known function given by

$$\begin{aligned}P(x, t, \tau_n) &= -Q_1\int_0^{\tau_n}\sin\left[\frac{x^2}{Q_2(t-\tau)}-\frac{\pi}{4}\right]\frac{d\tau}{\sqrt{(t-\tau)}}\\ &= -Q_1\Big[-2\sqrt{t-\tau_n}\sin\left(\frac{\pi z^2}{2}-\frac{\pi}{4}\right)\\ &\quad +2x\sqrt{\frac{2\pi}{Q_2}}\Big\{\frac{1}{\sqrt{2}}+f(z)\sin\left(\frac{\pi z^2}{2}-\frac{\pi}{4}\right)\\ &\quad -g(z)\cos\left(\frac{\pi z^2}{2}-\frac{\pi}{4}\right)\Big\}\Big]\end{aligned}$$

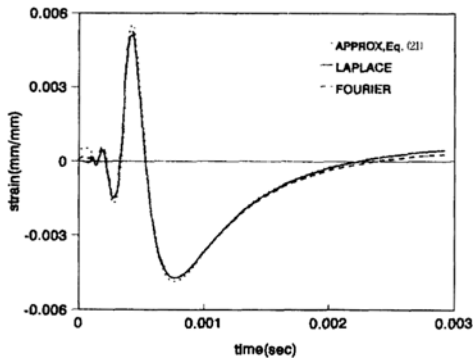
and

$$z=x\frac{\sqrt{2}}{\sqrt{\pi Q_2(t-\tau_n)}}$$

$$f(z)\approx\frac{1+0.926z}{2+1.792z+3.104z^2}$$

$$g(z)\approx\frac{1}{2+4.142z+3.492z^2+6.670z^3}$$

The solution of Eq. (21) is used as an approximate solution in Figs. 4(b) and 4(c). Figure 4 shows comparisons between the Laplace transform approach and the Fourier transform approach at three different locations. The numerical Laplace transform and the numerical Fourier transform algorithm are used with 10μs sampling time and 1024 sampling points. The Laplace transform approach yields better approximations than the Fourier transform approach. These results confirm the adequacy of the Laplace transform approach.



(b) x/h=100

Fig. 4 Strain histories at different monitoring positions

On the other hand, if the strain is measured at any location then it can be used to reconstruct the force history at the center, i. e. at the impact point, of the beam. The basic scheme to obtain the force is to take the strain history e(t), sample it every Δt, obtain the Laplace transform components $\bar{e}(\omega_j, x=0)$ and use Eq. (20) to obtain $\bar{F}(\omega_j)$. A frequency analysis requires a signal sample long enough to extract the harmonic content. However, since sample window are finite it is necessary to do a preprocessing of the strain data. Padding with zeros helps to insure that the signal is treated as a transient (Doyle, 1984a). Figure 5 shows the effect of two sample sizes.

In Fig. 5, S. P. denotes the number of sampling points. For force reconstruction a 1024-point numerical Laplace transform is used with a sampling time of 10μs and strain response of Eq. (2) is used as sampled data. The reconstructed force history exhibits a significant disturbance at the point of truncation. If the sample size is large

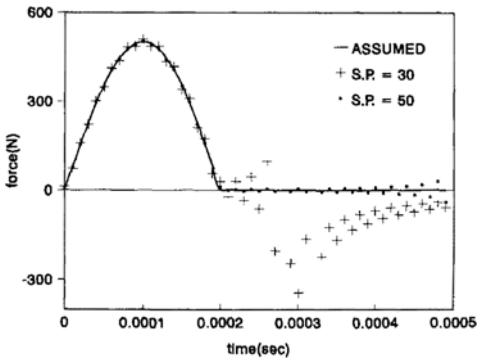


Fig. 5 The effect of sample size on the force history.

enough compared to the length of the force history, this problem is minimized.

5. Conclusions

A new force-strain relation for the Bernoulli-Euler beam is derived using the Laplace transform approach. The Laplace transform approach to dynamic strain response analysis and impact force reconstruction problems on an infinite beam can be simple, inexpensive and very accurate. Operating on the force-strain relation in the frequency domain allows signals for arbitrary position x to be handled much more conveniently than is the case in the time domain. The strain response analysis of the Laplace transform approach gives a significant improvement than that of Fourier transform approach. In practice, beam structures have a finite length. Eq. (10) should be generalized so as to include reflection effects from boundaries.

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